Math 206A Lecture 29 Notes

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December 7, 2018

1 Unfolding of Polytopes

1.1 Edge unfolding and Dürer's conejcture

You've probably seen an unfolding of a cube before. Think of the stencil you would need to make if you wanted to make an origami cube. How should we define this in general for other polytopes?

Definition 1.1. Let $P \subseteq \mathbb{R}^3$ be a polytope with $G(P) = (V, E)$. Let T be a spanning tree in G. An **edge unfolding** of P is $S = \partial P \setminus T$, isometric to a polygon in $\mathbb{R}^2$.

How do we know that when we unfold the polytope, it doesn't overlap with itself? In fact, it can.

Example 1.1. Take a cube, and remove a triangular pyramid from one of the corners (but close up the figure). Make an unfolding including cuts along two edges of the triangle left. When you unfold it, the triangle flap part will overlap with other faces of the polygon.

Here is a conjecture (which is still open).

Theorem 1.1 (Dürer, c. 1950). *For all P, there exists a spanning tree $T \subseteq G(P)$ such that $\partial P \setminus T$ has a non-overlapping unfolding.*

Many people have worked on this problem, but no one has proved it yet. It may not be true![1] In some sense, this theorem says that our original definition makes sense.

Theorem 1.2 (M. Ghomi, c. 2012). *For every polytope $P \subseteq \mathbb{R}^3$, there exists an affine transformation M such that MP has a non-overlapping unfolding.*

The idea of this proof is to use an affine transformation to stretch the affine polytope really thin.

[1] Maybe it's a good thing that this doesn't have a lot of practical applications.

1.2 The geodesic distance problem and source unfolding

Say we have 2 points on a cube. What is the shortest path on the cube from 1 point to the other? We can figure this out by looking at different unfoldings and taking the straight-line distance between the points. But this may be difficult if there are a lot of faces of your polytope; there could be a lot of possible unfoldings!

Theorem 1.3. *For all convex polytopes $P \subseteq \mathbb{R}^3$, if $S := \partial P$, then $|x, y|_S$ can be computed in polynomial time.*

The idea is called **source unfolding**. We'll discuss it using a cube. Fix $x \in S$, and let K be our **cut locus**. This is a set of points $K = \{z \in S : \exists \geq 2 \text{ shortest paths } z \to s\}$. Then no points on the face containing x are in K, and the edges with only 1 vertex touching this face containing x are completely contained in K. The idea is that no shortest path will intersect K. If we cut along K, we get an unfolding (but not necessarily an edge unfolding).

Here is the algorithm for this theorem:

1. Compute source unfolding at x (harder step)
2. Compute $|xy|_U$ (easier step)

How do we find the source unfolding? We use a continuous version of Dijkstra's algorithm.

Here is a conjecture:

Theorem 1.4. *For all d and convex $P \subseteq \mathbb{R}^d$, the number of cbonatorial shortest parths on ∂P is $n^{O(d^2)}$, where n is the number of facets of P.*

A combinatorial shortest path is a shortest path where we record the facets that the path passes through.

Theorem 1.5 (Miller-Pak)**.** *If the previous conjecture holds, then source unfolding for $P \subseteq \mathbb{R}^d$ for fixed d can be computed in polynomial time.*

This is a very technical result.[2]

[2] But it was hard to get published because everyone thought it should be trivial.